

POISSON-LIE T-DUALITY AND COURANT ALGEBROIDS

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1. INTRODUCTION

This note explains Poisson-Lie T-duality from the point of view of Courant algebroids. It contains basically nothing new: all the material is already contained in my letters [9] to Alan Weinstein written in 1998-99, which circulated in the “Poisson community” (including, among others, Anton Alekseev, Paul Bressler, Yvette Kosmann-Schwarzbach and Ping Xu) for some time.

During the 16 years since the letters were written, the basic technical tools (e.g. reduction of Courant algebroids [2]) were rediscovered and works linking (Abelian) T-duality and Courant algebroids appeared, notably the paper by Cavalcanti and Gualtieri [3]. I still decided to write my account and include details missing in [9]. Perhaps the most important reason is that I introduced exact Courant algebroids while trying to understand Poisson-Lie T-duality, and I believe that this duality, first introduced in [6], which generalized the usual Abelian T-duality, is essential for understanding of both Courant algebroids and of the world of T-dualities.

This note summarizes the first four letters of [9]. In particular, it doesn't deal with differential graded symplectic geometry and its link with Courant algebroids, which is discussed in the remaining letters. While it's certainly relevant for Poisson-Lie T-duality, I decided to exclude it to keep the focus on one thing, and also because I already wrote about it in [10].

2. EXACT COURANT ALGEBROIDS

Courant algebroids and Dirac structures were introduced by Liu, Weinstein and Xu in [8].

Definition 1. A Courant algebroid (CA) is a vector bundle $E \rightarrow M$ equipped with a non-degenerate quadratic form $\langle \cdot, \cdot \rangle$, with a bundle map

$$a : E \rightarrow TM$$

(the anchor map) and with a \mathbb{R} -bilinear map

$$[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \rightarrow \Gamma(E)$$

satisfying

$$A: [s, [t, u]] = [[s, t], u] + [t, [s, u]] \text{ for any } s, t, u \in \Gamma(E)$$

$$B: a([s, t]) = [a(s), a(t)] \text{ for any } s, t \in \Gamma(E)$$

$$C: [s, ft] = f[s, t] + (a(s)f)t \text{ for any } s, t \in \Gamma(E), f \in C^\infty(M)$$

$$D: a(s)\langle t, u \rangle = \langle [s, t], u \rangle + \langle t, [s, u] \rangle$$

$$E: [s, s] = a^t(d\langle s, s \rangle / 2), \text{ where } a^t : T^*M \rightarrow E^* \xrightarrow{\langle \cdot, \cdot \rangle} E \text{ is the transpose of } a.$$

A Dirac structure in E is a subbundle $L \subset E$ which is Lagrangian w.r.t $\langle \cdot, \cdot \rangle$ (i.e. $L^\perp = L$) such that $\Gamma(L)$ is closed under $[\cdot, \cdot]$.

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Remark 1. This definition from [9] is somewhat simpler than the (equivalent) original definition of Liu-Wenstein-Xu [8], who used the skew-symmetric part of $[\cdot, \cdot]$.

Axiom E can be replaced by the more innocent-looking

$$\langle s, [t, t] \rangle = \langle [s, t], t \rangle.$$

Axioms A–D are equivalent to the following: every section $s \in \Gamma(E)$ induces a vector field Z_s on E over $a(s)$, such that the flow of Z_s preserves all the structure; the bracket $[s, s']$ is the Lie derivative of s' under this flow. The map $s \mapsto Z_s$ is \mathbb{R} -linear. We have $[Z_s, Z_{s'}] = Z_{[s, s']}$ (as follows from axiom A).

Example 1 ([8]). If M is a point then E is a Lie algebra with invariant non-degenerate quadratic form $\langle \cdot, \cdot \rangle$.

Example 2 ([8]). If M is a manifold then $E = (T \oplus T^*)M$, with

$$(1a) \quad \langle (u, \alpha), (v, \beta) \rangle = \alpha(v) + \beta(u),$$

$$(1b) \quad a(u, \alpha) = u,$$

$$(1c) \quad [(u, \alpha), (v, \beta)] = ([u, v], L_u\beta - i_v d\alpha)$$

is the *standard* CA over M . In this case $Z_{(u,0)}$ is the natural lift of u to (the natural bundle) $(T \oplus T^*)M$, and $Z_{(0,\alpha)}$ is the vertical vector field with value $-i_v d\alpha$ at $(v, \beta) \in (T \oplus T^*)M$.

If $L \subset E$ is a Lagrangian vector subbundle of a CA (i.e. if $L^\perp = L$), we can measure the non-involutivity of L (i.e. its failure to be a Dirac structure) by

$$\mathcal{F}_L : \wedge^2 L \rightarrow E/L \cong L^*, \quad \mathcal{F}_L(s, t) = [s, t] \bmod L \quad (\forall s, t \in \Gamma(L))$$

where the isomorphism $E/L \cong L^*$ is given by $\langle \cdot, \cdot \rangle$, or equivalently by

$$\mathcal{H}_L \in \Gamma(\wedge^3 L^*), \quad \mathcal{H}_L(s, t, u) = \langle [s, t], u \rangle \quad (\forall s, t, u \in \Gamma(L))$$

(the fact that \mathcal{F}_L and \mathcal{H}_L are well-defined is readily verified; even though \mathcal{F}_L and \mathcal{H}_L are really the same object, it will be convenient to have a separate notation). L is a Dirac structure iff $\mathcal{F}_L = 0$ (or $\mathcal{H}_L = 0$).

If $E \rightarrow M$ is a CA with anchor map a then $a \circ a^t = 0$ (as follows from axioms E and B), i.e.

$$(2) \quad 0 \rightarrow T^*M \xrightarrow{a^t} E \xrightarrow{a} TM \rightarrow 0$$

is a chain complex.

Definition 2. A Courant algebroid $E \rightarrow M$ is exact if (2) is an exact sequence.

The simplest example of an exact CA is the standard CA; as we shall see below, every exact CA is locally standard.

Example 3. Let D be a Lie group and $G \subset D$ a closed subgroup. Let the Lie algebra \mathfrak{d} of D be equipped with a Ad -invariant non-degenerate quadratic form $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ such that $\mathfrak{g}^\perp = \mathfrak{g}$. Then $E = \mathfrak{d} \times D/G$ is an exact Courant algebroid over D/G . For constant sections of E the bracket is the Lie bracket on \mathfrak{d} and the anchor is the action of \mathfrak{d} on the homogeneous space D/G .

If $\mathfrak{h} \subset \mathfrak{d}$ is another Lagrangian Lie subalgebra, i.e. if \mathfrak{h} is a Dirac structure in \mathfrak{d} , then $\mathfrak{h} \times D/G \subset \mathfrak{d} \times D/G$ is a Dirac structure.

Exact CAs can be classified in the following way. If E is an exact CA then there is a Lagrangian subbundle $L \subset E$ such that $a|_L : L \rightarrow TM$ is an isomorphism (as can be seen by a partition of unity argument). We shall call such a subbundle $L \subset E$ a *connection* in E . Equivalently, a connection can be described as a splitting

$\sigma : TM \rightarrow E$ of the exact sequence (2), such that its image L is Lagrangian. Connections form an affine space over $\Omega^2(M)$ (if $\tau \in \Omega^2(M)$ then $(\tau + \sigma)(v) := \sigma(v) + a^t(i_v \tau)$).

If L is a connection then one can easily see that

$$H(u, v, w) := \langle [\sigma(u), \sigma(v)], \sigma(w) \rangle$$

defines a closed 3-form $H \in \Omega^3(M)$ (if we identify TM with L then $H = \mathcal{H}_L$); the 3-form H is called the *curvature* of the connection L . If we use $\sigma \oplus a^t$ to identify E with $TM \oplus T^*M$ then its anchor a and pairing \langle, \rangle are as in the standard CA, and the bracket is

$$(3) \quad [(u, \alpha), (v, \beta)] = ([u, v], L_u \beta - i_v d\alpha + H(u, v, \cdot)).$$

On the other hand, for any closed H the bracket (3) makes $TM \oplus T^*M$ to an exact Courant algebroid. If we change σ by a 2-form $\tau \in \Omega^2(M)$ then H gets replaced by $H + d\tau$. As a result, we have the following theorem:

Theorem 1 (classification of exact CAs). *Exact Courant algebroids over M are classified by $H^3(M, \mathbb{R})$; exact Courant algebroids with a chosen connection are classified by closed 3-forms H , with the bracket given by (3).*

If E is the exact CA given by (3), and $L \subset E$ a Dirac structure, then on each integral leaf $N \subset M$ of $a(L) \subset TM$ we have a 2-form α_N satisfying $d\alpha_N = H|_N$; the integral leaves and the 2-forms determine L uniquely.

3. EXACT CAs AND 2-DIMENSIONAL VARIATIONAL PROBLEMS

Let Σ be an oriented surface, M a manifold and $\omega \in \Omega^2(M)$ a 2-form. Let us consider the functional S on maps $f : \Sigma \rightarrow M$ given by

$$(4) \quad S[f] = \int_{\Sigma} f^* \omega.$$

We shall consider more general functionals in Remark 4 below; recall, however, that any local functional can be replaced by (4) if we replace M by an appropriate jet space (the de Donder-Weyl (=multisymplectic) method).

A map $f : \Sigma \rightarrow M$ is critical w.r.t. S iff

$$f^*(i_u d\omega) = 0$$

for every vector field u on M . If $\tau \in \Omega^2(M)$ is closed then ω and $\omega + \tau$ give equivalent variational problems. More generally, if $H \in \Omega^3_{cl}(M)$ is a closed 3-form, we can consider maps $f : \Sigma \rightarrow M$ satisfying

$$(5) \quad f^*(i_u H) = 0$$

and call them *critical* (or *H-critical*). As H is locally of the form $d\omega$, we can still see this equation as a solution of a variational problem.

Remark 2. From quantum point of view, to make the path integral formally meaningful, one needs to upgrade H to a Cheeger-Simons differential character, or equivalently to a class in the smooth Deligne cohomology [5].

Let us now consider the exact CA $E \rightarrow M$ with connection $L \subset E$ such that its curvature is H . (We can set $E = (T \oplus T^*)M$ with the bracket (3) and $L = TM$; if $H = d\omega$ we can equivalently take the standard CA and set L to be the graph of $\omega : TM \rightarrow T^*M$). For any map $f : \Sigma \rightarrow M$ let

$$\tilde{T}f : T\Sigma \rightarrow L$$

be the lift of the tangent map $Tf : T\Sigma \rightarrow TM$ given by $a \circ \tilde{T}f = Tf$. The map $\tilde{T}f$ can be used to pull back sections of $\bigwedge L^*$ to differential forms on Σ ; this pullback will be denoted by f^* .

The Euler-Lagrange equation (5) can be rephrased as a ‘zero-curvature condition’.

Proposition 1. *A map $f : \Sigma \rightarrow M$ is critical iff*

$$f^*(\mathcal{F}_L) = 0 \quad (\in \Omega^2(\Sigma, f^*(E/L))).$$

The importance of E is that its sections, rather than just vector fields on M , can be interpreted as symmetries and give rise to conservation laws.

Theorem 2 (“Noether”). *If $s \in \Gamma(E)$ is such that the flow of Z_s preserves L then for any critical map $f : \Sigma \rightarrow M$ the 1-form $f^*\langle s, \cdot \rangle \in \Omega^1(\Sigma)$ satisfies*

$$d(f^*\langle s, \cdot \rangle) = 0.$$

Proof. We identify E with $(T \oplus T^*)M$ using the connection L ; the bracket on E is then (3) and $L = TM$. If $s = (u, \alpha)$ then $f^*\langle s, \cdot \rangle = f^*\alpha$. The flow of Z_s preserves L iff

$$d\alpha + i_u H = 0.$$

If f is critical then $f^*(i_u H) = 0$. We thus get $d(f^*\langle s, \cdot \rangle) = f^*(d\alpha) = 0$. \square

The main theme of this paper is the study of symmetries that in place of closed 1-forms give rise to flat connection. The fact that Euler-Lagrange equations can be seen as a zero-curvature condition (Proposition 1) will play an important role. To explain it we will need equivariant exact CAs, which we introduce in the following section.

Remark 3. If we considered 1-dimensional variational problems instead of 2-dimensional then exact CAs would be replaced by Lie algebroids $A \rightarrow M$ which are extensions $0 \rightarrow \mathbb{R} \rightarrow A \rightarrow TM \rightarrow 0$. A splitting of this extension, i.e. a connection in A , gives rise to a closed 2-form (the curvature of the connection). To make formal sense of the path integral we would rather need a principal $U(1)$ -bundle $P \rightarrow M$ with a connection; in this case $A = (TP)/U(1)$.

In the case of 2-dimensional problems principal $U(1)$ -bundles are replaced by $U(1)$ -gerbes (as observed by Brylinski [1], reinterpreting Gawędzki’s approach via smooth Deligne cohomology [5]). Exact CAs are thus closely related to $U(1)$ -gerbes.

Remark 4. If Σ is a surface with pseudo-conformal structure, with local light-like coordinates t_1, t_2 , and $r \in \Gamma(T^{*\otimes 2}M)$ is a tensor field on M , let us consider the standard σ -model action functional on maps $f : \Sigma \rightarrow M$,

$$S_r[f] = \int_{\Sigma} r \left(\frac{\partial f}{\partial t_1}, \frac{\partial f}{\partial t_2} \right) dt_1 dt_2.$$

Let E be the standard CA and $R \subset E$ be the graph of $TM \rightarrow T^*M$, $v \mapsto r(v, \cdot)$ (then R^\perp is the graph of $v \mapsto -r(\cdot, v)$). For any $f : \Sigma \rightarrow M$ let us lift $Tf : T\Sigma \rightarrow TM$ to $\tilde{T}f : T\Sigma \rightarrow E$ by requiring $\tilde{T}f(\partial_{t_1}) \in R$ and $\tilde{T}f(\partial_{t_2}) \in R^\perp$. In this case Noether theorem says:

If R is invariant under the flow of Z_s for some $s \in \Gamma(E)$, and if $f : \Sigma \rightarrow M$ is critical for the functional S_r , then $d(f^\langle s, \cdot \rangle) = 0$.*

Proposition 1 becomes trickier and we don’t state it here (see [10, Section 5], where it is formulated in terms of differential graded manifolds). A similar picture can be found for any Lagrangian density depending on the first derivatives of f .

4. EQUIVARIANT COURANT ALGEBROIDS AND THEIR REDUCTION

Let \mathfrak{g} be a Lie algebra, $E \rightarrow M$ a Courant algebroid, and $\rho : \mathfrak{g} \rightarrow \Gamma(E)$ a $[\cdot, \cdot]$ -preserving linear map. The functions $\langle \rho(\xi), \rho(\eta) \rangle \in C^\infty(M)$, $\xi, \eta \in \mathfrak{g}$, are constant (provided M is connected), as

$$0 = \rho([\xi, \eta] + [\eta, \xi]) = [\rho(\xi), \rho(\eta)] + [\rho(\eta), \rho(\xi)] = d\langle \rho(\xi), \rho(\eta) \rangle.$$

The resulting (possibly degenerate) pairing $\langle \rho(\xi), \rho(\eta) \rangle$ on \mathfrak{g} is ad -invariant. This leads us to the following definition.

Definition 3. Let \mathfrak{g} be a Lie algebra and $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ an invariant symmetric bilinear pairing on \mathfrak{g} (possibly degenerate). If E is a CA, a representation of $(\mathfrak{g}, \langle \cdot, \cdot \rangle_{\mathfrak{g}})$ in E is a linear map $\rho : \mathfrak{g} \rightarrow \Gamma(E)$ such that $\rho([\xi, \eta]) = [\rho(\xi), \rho(\eta)]$ and $\langle \rho(\xi), \rho(\eta) \rangle = \langle \xi, \eta \rangle_{\mathfrak{g}}$.

A representation of \mathfrak{g} in E gives us an action of \mathfrak{g} on E by $Z_{\rho(\xi)}$'s. If G is a connected Lie group with the Lie algebra \mathfrak{g} , and the action of \mathfrak{g} on E gives rise to an action of G in E , we shall say that A is a G -equivariant CA.

Remark 5. If exact CAs $E \rightarrow M$ are seen as approximations of $U(1)$ -gerbes over M then the “gerby” version of a G -equivariant exact CA $E \rightarrow M$ is a multiplicative gerbe over G acting on a gerbe over M . In this context it's best to replace exact CAs $E \rightarrow M$ with principal $\mathbb{R}[2]$ -bundles $X \rightarrow T[1]M$ in the category of differential graded manifolds. Multiplicative gerbes over G are approximated by central extensions of the group $T[1]G$ by $\mathbb{R}[2]$, and such extensions are classified by invariant symmetric bilinear forms on \mathfrak{g} . See [10, section 3] for some details.

It is easy to see that \mathfrak{g} -invariant sections of E , orthogonal to the image of \mathfrak{g} in E , are closed under the bracket $[\cdot, \cdot]$. This gives us, after we mod out by the sections which are in the kernel of $\langle \cdot, \cdot \rangle$, the following theorem.

Theorem 3 (reduction of CAs). Let $E \rightarrow M$ be an G -equivariant CA. Suppose that the action of G on M is free and proper. For any $x \in M$ let

$$(E/G)_x = (\rho_x(\mathfrak{g}))^\perp / (\rho_x(\mathfrak{g})^\perp \cap \rho_x(\mathfrak{g})) = (\rho_x(\mathfrak{g}))^\perp / \rho_x(\mathfrak{g}'),$$

where $\mathfrak{g}' \subset \mathfrak{g}$ is the kernel of $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$. After factorization by the action of G , E/G becomes a vector bundle over M/G . The CA structure on E descends to a CA structure on E/G . If E is exact and $\langle \cdot, \cdot \rangle_{\mathfrak{g}} = 0$ (and thus $\mathfrak{g}' = \mathfrak{g}$) then E/G is exact.

Remark 6. This reduction procedure was rediscovered and further generalized in [2].

If $M \rightarrow M/G$ is a principal G -bundle (i.e. if G acts freely and properly on M), let its Pontryagin class be

$$[\langle F, F \rangle_{\mathfrak{g}}] \in H^4(M, \mathbb{R})$$

where F is the curvature of a connection on the principal bundle.

Theorem 4 (classification of equivariant exact CAs). If G acts freely and properly on M then M admits a G -equivariant exact CA iff the Pontryagin class of the principal G -bundle $M \rightarrow M/G$ vanishes.

There is a natural free and transitive action of the group $H^3(M/G, \mathbb{R})$ on the set of isomorphism classes of G -equivariant CAs $E \rightarrow M$, where the class of a closed 3-form $\gamma \in \Omega_{cl}^3(M/G)$ acts by modifying the bracket on $E \rightarrow M$ via

$$[s, t]_{new} = [s, t] + a^t(p^*\gamma(a(s), a(t), \cdot)).$$

Proof. Let us choose a connection $A \in \Omega^1(M, \mathfrak{g})$ on the principal G -bundle $M \rightarrow M/G$. If $E \rightarrow M$ is a G -equivariant exact CA, we can choose a G -invariant Lagrangian splitting $E \cong (T \oplus T^*)M$ (i.e. a connection $L \subset E$), such that

$$(6) \quad \rho(\xi) = (\xi_M, \frac{1}{2} \langle \xi, A \rangle_{\mathfrak{g}}) \in \Gamma((T \oplus T^*)M) \quad (\forall \xi \in \mathfrak{g}),$$

where $\xi_M = a(\rho(\xi))$ is the vector field on M given by $\xi \in \mathfrak{g}$. The bracket on $E \cong (T \oplus T^*)M$ is now given by (3) for some G -invariant closed 3-form $H \in \Omega_{cl}^3(M)$.

G -invariance of the splitting $E \cong (T \oplus T^*)M$ means that for every vector field u on M and any $\xi \in \mathfrak{g}$ the section $[\rho(\xi), (u, 0)]$ of E is a vector field (i.e. its 1-form part is zero). Equivalently

$$(7) \quad i_{\xi_M} H = -\frac{1}{2} \langle \xi, dA \rangle_{\mathfrak{g}}.$$

Equation (7) also ensures that ρ is a representation of \mathfrak{g} in E .

The Chern-Simons 3-form $\mathbf{cs} = \langle A, dA \rangle_{\mathfrak{g}} + \frac{1}{3} \langle [A, A], A \rangle_{\mathfrak{g}}$ satisfies (up to a factor) the same equation

$$i_{\xi_M} \mathbf{cs} = \langle \xi, dA \rangle_{\mathfrak{g}},$$

and $d\mathbf{cs} = p^* \langle F, F \rangle_{\mathfrak{g}}$, where $p : M \rightarrow M/G$ is the projection and F the curvature of A . As a result, the general solution of (7) is

$$H = p^* \theta - \mathbf{cs}/2, \quad d\theta = \frac{1}{2} \langle F, F \rangle_{\mathfrak{g}}.$$

If we change the splitting of E by a 2-form $\tau \in \Omega^2(M/G)$ then H gets replaced by $H + p^* d\tau$, i.e. θ by $\theta + d\tau$. G -equivariant CAs over M are thus classified by solutions of $d\theta = \frac{1}{2} \langle F, F \rangle_{\mathfrak{g}}$ modulo exact 3-forms. \square

As an application of the reduction procedure, let us now describe a construction/classification of transitive CAs, i.e. of CAs with surjective anchors. If $\tilde{E} \rightarrow N$ is a transitive CA with anchor $\tilde{a} : \tilde{E} \rightarrow TN$ then $B := \tilde{E}/\text{Im } \tilde{a}^t$ is a transitive Lie algebroid with an invariant inner product on the bundle of vertical Lie algebras.

Theorem 5 (exact equivariant vs. transitive CAs). *If $M \rightarrow N = M/D$ is a principal D -bundle and $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ is non-degenerate then the reduction by D gives an equivalence between D -equivariant exact CAs $E \rightarrow M$ and transitive CAs $\tilde{E} \rightarrow N$ such that $\tilde{E}/\text{Im } \tilde{a}^t = (TM)/D$.*

Proof. If $E \rightarrow M$ is a D -equivariant CA then the fact that $\tilde{E} := E/D$ is transitive and $\tilde{E}/\text{Im } \tilde{a}^t = (TM)/D$ follows from the definition of E/D .

If $\tilde{E} \rightarrow N$ is a transitive CA such that $\tilde{E}/\text{Im } \tilde{a}^t = (TM)/D$ then we can (re)construct a D -equivariant exact $E \rightarrow M$ as follows. Let

$$E := p^* \tilde{E} \oplus \mathfrak{d},$$

with the following structure: the anchor

$$a_E : p^* \tilde{E} \oplus \mathfrak{d} \rightarrow TM$$

is the sum of the projection $p^* \tilde{E} \rightarrow p^* B = TM$ and of the natural map $\mathfrak{d} \times M \rightarrow TM$, the pairing $\langle \cdot, \cdot \rangle_E$ is the direct sum of the pairings on \tilde{E} and on \mathfrak{d} , and the bracket is given by

$$[p^* s, p^* t]_E = p^* [s, t]_{\tilde{E}}, \quad [\xi, \eta]_E = [\xi, \eta]_{\mathfrak{d}}, \quad [p^* s, \xi]_E = 0$$

for all $s, t \in \Gamma(\tilde{E})$, $\xi, \eta \in \mathfrak{d}$. \square

Remark 7. If $B \rightarrow N$ is an arbitrary transitive Lie algebroid with invariant inner product on its vertical Lie algebras then transitive CAs $\tilde{E} \rightarrow N$ such that $\tilde{E}/\tilde{a}^t = B$ exist iff the Pontryagin class of B vanishes; in this case $H^3(N, \mathbb{R})$ acts freely and transitively on their isomorphism classes (isomorphisms which are identity on B). If $B = (TM)/D$ as above then this result follows from Theorems 4 and 5. For general B it can be proven by a direct calculation; we don't need this result here, so we refer the reader to [9, no.4]. This result was rediscovered and extended to regular CAs in [4].

5. REDUCTION AND CURVATURE

In this section \mathfrak{d} is a Lie algebra with a non-degenerate invariant symmetric pairing $\langle \cdot, \cdot \rangle_{\mathfrak{d}}$ and D a connected Lie group with Lie algebra \mathfrak{d} .

Let D act freely and properly on a manifold M and let $E \rightarrow M$ be an equivariant CA. Let $E/D \rightarrow M/D$ be the reduction of E ; we have $p_D^* E/D = \rho(\mathfrak{d})^\perp \subset E$, where $p_D : M \rightarrow M/D$ is the projection.

Let $L_D \subset E/D$ be a Lagrangian subbundle. Then

$$L := p_D^* L_D \subset \rho(\mathfrak{d})^\perp \subset E$$

is a D -invariant Lagrangian subbundle of $\rho(\mathfrak{d})^\perp$; any D -invariant Lagrangian subbundle of $\rho(\mathfrak{d})^\perp$ is of this form.

We define

$$\mathcal{F}_L : \bigwedge^2 L \rightarrow (\rho(\mathfrak{d})^\perp / L) \cong L^*, \quad \mathcal{F}_L(s, t) = [s, t] \bmod L \quad (\forall s, t \in \Gamma(L))$$

and

$$\mathcal{H}_L \in \Gamma(\bigwedge^3 L^*), \quad \mathcal{H}_L(s, t, u) = \langle [s, t], u \rangle \quad (\forall s, t, u \in \Gamma(L))$$

(a quick inspection shows that \mathcal{F}_L and \mathcal{H}_L are well defined). Notice (by using D -invariant sections in the definition of \mathcal{F}_L and \mathcal{H}_L) that

$$\mathcal{F}_L = p_D^* \mathcal{F}_{L_D} \text{ and } \mathcal{H}_L = p_D^* \mathcal{H}_{L_D}.$$

Let now $G \subset D$ be a Lagrangian subgroup (i.e. its Lie algebra \mathfrak{g} is a Lagrangian subspace of \mathfrak{d} , or equivalently $(\mathfrak{d}, \mathfrak{g})$ is a Manin pair). Let us consider the reduced CA $E/G \rightarrow M/G$. We have a natural identification $E/G \cong \rho(\mathfrak{d})^\perp / G$ (as $\rho(\mathfrak{g})^\perp = \rho(\mathfrak{g}) \oplus \rho(\mathfrak{d})^\perp$ and thus $\rho(\mathfrak{g})^\perp / \rho(\mathfrak{g}) \cong \rho(\mathfrak{d})^\perp$). Let us define a Lagrangian subbundle $L_G \subset E/G$ to be $L_G = L/G$ (i.e. $L = p_G^* L_G$). By using G -invariant sections of L we get

$$(8) \quad \mathcal{F}_L = p_G^* \mathcal{F}_{L_G} \text{ and } \mathcal{H}_L = p_G^* \mathcal{H}_{L_G},$$

where $p_G : M \rightarrow M/G$ is the projection. As a result we have

Proposition 2. $p_D^* \mathcal{F}_{L_D} = p_G^* \mathcal{F}_{L_G}$ and $p_D^* \mathcal{H}_{L_D} = p_G^* \mathcal{H}_{L_G}$. In particular, $L_G \subset E/G$ is a Dirac structure iff $L_D \subset E/D$ is a Dirac structure.

Let us now suppose in addition that E is exact (which implies that E/G is exact) and that its anchor $a : E \rightarrow TM$ is injective on $L \subset E$. Let

$$V := a(L) \subset TM.$$

Notice that

$$\text{rank } V = \text{rank } L = \dim M - \frac{1}{2} \dim \mathfrak{d} = \dim M/G.$$

The non-involutivity of the distribution $V \subset TM$ is measured by its curvature

$$F_V : \bigwedge^2 V \rightarrow TM/V, \quad F_V(u, v) = [u, v] \bmod V \quad (\forall u, v \in \Gamma(V)).$$

From definitions we get that

$$(9) \quad F_V(a(s), a(t)) = a(\mathcal{F}_L(s, t)) \quad \forall s, t \in \Gamma(L).$$

6. NON-ABELIAN CONSERVATION LAWS AND POISSON-LIE T-DUALITY

Poisson-Lie T-duality is a geometric version of “non-Abelian Noether theorem”, where a symmetry gives rise to a flat connection instead of a closed 1-form, and also an equivalence between two (or more) variational problems. It was introduced in [6]. The idea of exact CAs was extracted from this T-duality; the following is a “coordinate-free” interpretation of Poisson-Lie T-duality in terms of exact CAs.

Let us use the setup and notation of the previous section: $E \rightarrow M$ is a D -equivariant exact CA (the action of D on M is free and proper), $L \subset \rho(\mathfrak{d})^\perp$ is a

Lagrangian D -invariant subbundle such that the anchor a is injective on L , and $G \subset D$ is a Lagrangian subgroup.

The Lagrangian subbundle $L_G \subset E/G$ is a connection iff $V := a(L) \subset TM$ is transverse to the fibers of the projection $M \rightarrow M/G$. Supposing this (or removing the points where transversality fails), let

$$H_G \in \Omega_{cl}^3(M/G)$$

be the curvature of the connection $L_G \subset E/G$ and let $A_G \in \Omega^1(M, \mathfrak{g})$ be the connection on the principal G -bundle $p_G : M \rightarrow M/G$ whose horizontal bundle is $V \subset TM$.

Theorem 6 (“non-Abelian Noether”). *If $f : \Sigma \rightarrow M/G$ is H_G -critical then f^*A_G is a flat connection on the principal G -bundle $f^*M \rightarrow \Sigma$.*

Proof. It follows immediately from Proposition 1 and relations (8) and (9) \square

Motivated by Proposition 1, we shall say that a map $\phi : \Sigma \rightarrow M$ is L -critical if the tangent map $T\phi : T\Sigma \rightarrow TM$ can be lifted to a vector bundle map $\tilde{T}\phi : T\Sigma \rightarrow L$ (i.e. if the image of $T\phi$ is in $V = a(L)$) and if $\phi^*\mathcal{F}_L = 0 \in \Omega^2(\Sigma, \phi^*L^*)$. Notice that the action of D on M sends L -critical maps $\Sigma \rightarrow M$ to L -critical maps; by the following theorem, such maps are equivalent to H_G -critical maps $\Sigma \rightarrow M/G$.

Theorem 7 (Poisson-Lie T-duality). *If $\phi : \Sigma \rightarrow M$ is L -critical then $p_G \circ \phi : \Sigma \rightarrow M/G$ is H_G -critical. If Σ is 1-connected and $f : \Sigma \rightarrow M/G$ is H_G -critical then there is a L -critical map $\phi : \Sigma \rightarrow M$ such that $f = p_G \circ \phi$; the map ϕ is unique up to the action of G .*

If $G' \subset D$ is another Lagrangian subgroup then lifting H_G -critical maps to L -critical and projecting them to M/G' gives us an equivalence between H_G -critical maps $\Sigma \rightarrow M/G$ and $H_{G'}$ -critical maps $\Sigma \rightarrow M/G'$.

Proof. If $\phi : \Sigma \rightarrow M$ is L -critical then H_G -criticality of $p_G \circ \phi$ follows from Proposition 1 and from (8). If Σ is 1-connected and $f : \Sigma \rightarrow M/G$ is H_G -critical then by Theorem 6 there is a map $\phi : \Sigma \rightarrow M$ such that $p_G \circ \phi = f$ and such that the image of $T\phi$ is in V (ϕ is unique up to the action of G). Relation (8) and H_G -criticality of f then imply that ϕ is L -critical. \square

The name “Poisson-Lie T-duality” comes from the case when $\mathfrak{g} \cap \mathfrak{g}' = 0$, i.e. when $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}')$ is a Manin triple and G and G' a dual pair of Poisson-Lie groups.

Remark 8. If we start with a half-dimensional subbundle $R_D \subset E/D$ in place of L_D (we don’t suppose that R_D is Lagrangian) we obtain a subbundle $R_G \subset E/G$. When we locally trivialize the exact CA E/G , we get a σ -model as described in Remark 4. Theorems 6 and 7 remain valid (after the appropriate modification). It was for this type of models that Poisson-Lie T-duality was originally formulated in [6] (without using the language of CAs). We don’t give details, as it’s easier to pass to equivalent σ -models given by 2-forms / closed 3-forms.

This picture was used in [3] in the case of Abelian D (without discussing critical maps etc.).

7. DIRAC STRUCTURES AND BOUNDARY CONDITIONS (D-BRANES)

Let us return to variational problems. Let M be a manifold, $N \subset M$ a submanifold, and let us choose forms $\omega \in \Omega^2(M)$, $\alpha_N \in \Omega^1(N)$. If Σ is a surface, let us consider the action functional

$$S[f] = \int_{\Sigma} f^*\omega + \int_{\partial\Sigma} f^*\alpha_N$$

defined on maps $f : \Sigma \rightarrow M$ mapping $\partial\Sigma$ to N . The Euler-Lagrange equations for critical f 's is now

$$f^*i_u d\omega = 0 \text{ on } \Sigma, \quad f^*i_v(\omega|_N + d\alpha_N) = 0 \text{ on } \partial\Sigma$$

for every vector fields u on M and v on N .

More invariantly and generally, we choose a closed 3-form $H \in \Omega_{cl}^3(M)$ and a 2-form $\beta_N \in \Omega^2(N)$ such that $d\beta_N = H|_N$. Locally then we can find ω and α_N such that $H = d\omega$ and $\beta_N = \omega|_N + d\alpha_N$. The Euler-Lagrange relations now say

$$(10) \quad f^*i_u H = 0 \text{ on } \Sigma, \quad f^*i_v \beta_N = 0 \text{ on } \partial\Sigma.$$

Remark 9. For quantization, to make formal sense of the path integral, the pair (H, β_N) should be extended to a relative Cheeger-Simons differential character. This fact was discussed in the case of the WZW-model in [7] and in full generality in [11].

As in Section 3 let $L \subset E$ be the exact CA with connection whose curvature is H . Let $C \subset E$ be a Dirac structure. On any leaf $N \subset M$ of the distribution $a(C) \subset TM$ we then have a 2-form $\beta_N \in \Omega^2(N)$ such that $d\beta_N = H|_N$. We can thus use the Dirac structure C to impose a boundary condition; the map f is required to send each component of $\partial\Sigma$ to a leaf N , and critical maps are given by the Euler-Lagrange equation (10).

Proposition 1 has now the following form.

Proposition 3. *A map $f : \Sigma \rightarrow M$ is critical with the boundary condition given by C iff*

$$f^*(\mathcal{F}_L) = 0 \quad (\in \Omega^2(\Sigma, f^*(E/L)))$$

and

$$(\tilde{T}f)(T(\partial\Sigma)) \subset C.$$

Let us now describe Dirac structures (and thus boundary conditions) compatible with Poisson-Lie T-duality. We shall use the setup of Section 6: a free and proper action of D on M , a D -equivariant exact CA $E \rightarrow M$, a Lagrangian D -invariant subbundle $L \subset \rho(\mathfrak{d})^\perp$ such that a is injective on L , a Lagrangian subgroup $G \subset D$. This data gives us the connection L_G in the exact CA $E_{/G} \rightarrow M/G$ with curvature $H_G \in \Omega_{cl}^3(M/G)$.

We can now use Proposition 2 to describe Dirac structures (or boundary conditions) in M/G compatible with Poisson-Lie T-duality. We start with a Dirac structure $C_D \subset E_{/D}$; by Proposition 2 (using “ C ” in place of “ L ”) it gives us a Dirac structure $C_G \subset E_{/G}$ and a D -invariant subbundle $C \subset \rho(\mathfrak{d})^\perp \subset E$.

If $f : \Sigma \rightarrow M/G$ is a H_G -critical map satisfying the boundary condition given by C_G then its lift $\phi : \Sigma \rightarrow M$ (see Theorem 7) will satisfy

$$\tilde{T}\phi(T(\partial\Sigma)) \subset C$$

and thus, if $G' \subset D$ is another Lagrangian subgroup, the map $p_{G'} \circ \phi : \Sigma \rightarrow M/G'$ will be $H_{G'}$ -critical (Theorem 7) and will satisfy the boundary condition given by $C_{G'} \subset E_{/G'}$.

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